On a universal best choice algorithm for partially ordered sets

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Abstract

For the only known universal best choice algorithm for partially ordered sets with known cardinality and unknown order (proposed by J. Preater) we improve the estimation of the lower bound of its chance of success from the hitherto known constant 1/8 to 1/4. We also show that this result is the best possible for this algorithm, i.e., the 1/4 bound cannot be further improved.

1. Introduction. We consider the following selection process. A selector knows the cardinality $|X| = n$ of some set X which is equipped with a partial order $\prec$. The selector examines the elements of X one by one: $\pi(1), \pi(2), \ldots, \pi(n)$, where $(\pi(1), \pi(2), \ldots, \pi(n))$ is any of all equiprobable permutations of X. The permutation π is unknown to the selector. Object $\pi(t)$ appears at time $t \in \{1, 2, \ldots, n\}$. The selector can compare all the objects that have been examined so far. Thus at the time t the selector knows only the relation $\{(i,j) : i, j \leq t,\, \pi(i) \prec \pi(j)\}$. The selector's aim is to stop the process at some moment $t = \tau(\pi)$ in such a way that the currently examined element $\pi(t) = \pi(\tau(\pi))$ be maximal with respect to $\prec$. Thus the decision is an on-line one, i.e., the elements examined at times $i < t$ are not available any more and the knowledge of the selector at the moment of the choice is only that about the present object and the previous ones. Such a random variable τ is called a *stopping time*.

If $\prec$ is linear and $|X| = n$ is known to the selector, the above on-line selection problem is called the "secretary problem" due to the interpretation of elements of X as candidates for a job of a secretary, examined one by one by an administrator who is to choose on-line the candidate that would be the best of all. For the solution $\tau^* : \pi \mapsto \tau^*(\pi) \in \{1, \ldots, n\}$ of this linear problem, maximizing $\mathbf{Pr}[\pi(\tau) = \max X]$ over all stopping times τ, consult [3]. This probability tends to $\frac{1}{e}$ as $n \to \infty$. The reader can consult the comprehensive article [1] for information on the linear case.

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Examining models where the selector chooses from a partially ordered set has good justification as far as possible applications are concerned. Namely, in real life, frequently the selection is made from elements that are not linearly ordered. The optimal stopping time τ^* was given in the case where $\prec$ is known to the selector as the order of the complete binary tree of fixed depth k (see [4]). In this case the probability $\mathbf{Pr}[\pi(\tau^*) = \max X]$ tends to 1 as $k \to \infty$. In [2] the reader can find a comprehensive account of several articles concerning threshold type strategies when the elements do not form a linear order. For the best choice from partially ordered objects see also [7].

Here we are concerned with a surprising result of a new nature that was obtained by J. Preater [5,6] ([6] is a shorter published version of the preprint [5]). In the general partial order case, where $|X| = n$ is known and $\prec$ is not known to the selector, a universal randomized stopping time was defined ensuring the selector's success with uniformly positive probability (i.e., always not smaller than some positive $\delta > 0$), namely at least $\frac{1}{8}$. Actually, two algorithms were mentioned in [5]: Policy A in [5,6] and Policy B in [5], the latter one stronger than the former one and being, actually, its obvious modification; but the estimation from below of the probability of the best choice for both of them remained $\frac{1}{8}$. Here (Theorem 3.1, below) we show that Preater's Policy B gives (always) the probability of success not smaller than $\frac{1}{4}$. We also show that this estimation for this algorithm cannot be improved (Theorem 3.2).

Policy B is carried out in three steps:

1. Flip a symmetric coin n times. If it comes down tails M times, examine the first M objects of a random permutation of X. They form a set Y. State the height of the set Y, i.e., the maximum length of a chain consisting of elements of Y; denote it by $h(Y)$.

2. Flip a symmetric coin once. If it comes down heads, tag all the objects from Y whose induced height is equal to $h(Y)$; if it comes down tails, tag all the objects from Y whose induced height is greater than or equal to $h(Y) - 1$.

3. Examine further the remaining elements of X. Pick the first object that dominates any of the tagged ones and is maximal up to now. If such an object does not appear, pick the final one.

(For example, compare Fig. 1–4; here $n = 15$, $\pi(1), \dots, \pi(15)$ is the permutation of X that the selector deals with, the order of X is depicted in Fig. 1; here we assume $M = 6$, therefore $Y = \{\pi(1), \dots, \pi(6)\}$, and, at time $t = 6$, the selector can see Y with the order inherited from X, it appears in Fig. 2; if the coin comes down tails the selector tags the elements that are in squares in Fig. 3, namely $\pi(2)$, $\pi(3)$ and $\pi(6)$; if the coin comes down heads the selector tags the elements that are in squares in Fig. 4, namely $\pi(2)$ and $\pi(6)$.)

The first step of the algorithm, which involves flipping the coin $n = |X|$ times, is to ensure that each particular element of X belongs to Y with probability $1/2$ independently of the membership in Y of the other elements. The second step of the algorithm, which involves flipping the coin once, is typical for randomized algorithms and makes a random choice between cases, allowing for situations where some (but not all) of the cases are unlikely to lead to success. The way elements are tagged ensures that in sufficiently many situations the elements above the tagged ones are guaranteed to be maximal.

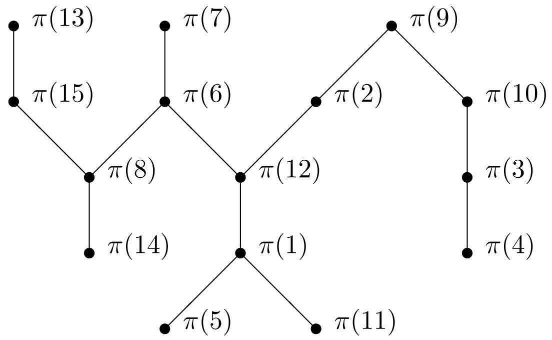

Fig. 1

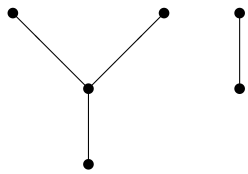

Fig. 2

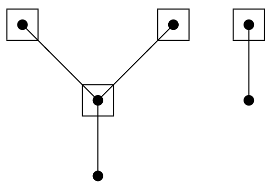

Fig. 3

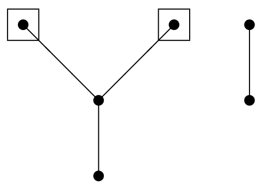

Fig. 4

2. Notation and Formal Model. All partial orders considered in this paper are assumed to be strict.

For any set X, let $\mathbf{P}(X) = \{V : V \subseteq X\}$, and let $|X|$ denote the number of elements of X. For a partially ordered set $(X, \prec)$ and an element $x \in X$, let $L_{(X,\prec)}(x)$ denote the set of all predecessors of x, i.e., $L_{(X,\prec)}(x) = \{z \in X : z \prec x\}$. Let $U_{(X,\prec)}(x)$ denote the set of all successors of x, i.e., $U_{(X,\prec)}(x) = \{z \in X : x \prec z\}$. If $(X, \prec)$ is clear from the context we shall simply write $L(x)$, $U(x)$, respectively.

Let $S_1(X)$ denote the set of all maximal elements of X. Let $S_2(X)$ denote the set of elements of $X \setminus S_1(X)$ with successors only in $S_1(X)$. Let $S_3 = X \setminus (S_1(X) \cup S_2(X))$. For $Y \subseteq X$, let $\operatorname{Max} Y$ denote the set of all maximal elements of Y with respect to the restricted partial order $\prec \cap Y^2$. In the special case of $Y = X$, of course, $\operatorname{Max} X = S_1(X)$. For $y \in Y \subseteq X$, let $h_Y(y)$ denote the maximum length of a chain consisting of elements of Y with greatest element equal to y. This number is called *the height of* y in Y. Let $h(Y) = \max\{h_Y(y) : y \in Y\}$. This number is called *the height of* Y. Let

$$Y_i = \{y \in Y : h_Y(y) \geq h(Y) - i\}.$$

Let us also introduce the following notion. For $T \subseteq X$, we define T^+ as

$$T^+ = \{x \in X \setminus T : (\exists\, v \in T)(v \prec x) \text{ and } (\forall\, w \in T)(x \not\prec w)\},$$

the idea being that, if T is the set of objects tagged at stage 2 of Policy B, then at stage 3 the policy will pick an object in T^+ (unless T^+ is empty, in which case the final object is picked).

Let us also introduce the following class $\mathcal{P}$ of ranked partially ordered sets of height 2. Namely:

$$\mathcal{P} = \{(P, \prec) : h(P) = 2 \text{ and } (\forall x \in \operatorname{Max} P)(h_P(x) = 2)\}.$$

Let us fix the notation of some important partial orders $(P, \prec)$ in $\mathcal{P}$. In the definitions below, we assume that all elements listed as members of P are pairwise different.

We say that $(P, \prec)$ *is of type* $\mathbf{N}$ (Fig. 5), if

$$P = \{x_1, x_2, z_1, z_2\}$$

and

$$\prec = \{(z_1, x_1), (z_2, x_1), (z_2, x_2)\}.$$

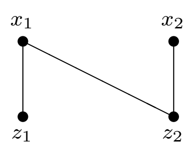

Fig. 5

We say that $(P, \prec)$ *is of type* $\mathbf{V}^{(k)}$ (Fig. 6), $k \geq 1$, if

$$P = \{z, x_1, \ldots, x_k\}$$

and

$$\prec = \{(z, x_1), \ldots, (z, x_k)\}.$$

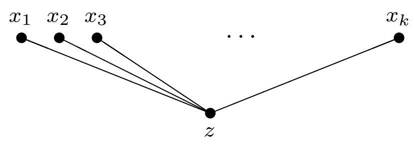

Fig. 6

We say that $(P, \prec)$ *is of type* $\mathbf{\Lambda}_{(k)}$ (Fig. 7), $k \geq 1$, if

$$P = \{x, z_1, \dots, z_k\}$$

and

$$\prec = \{(z_1, x), \dots, (z_k, x)\}.$$

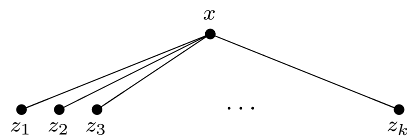

Fig. 7

We say that $(P, \prec)$ *is of type* $\mathbf{I}$, if $(P, \prec)$ is of type $\mathbf{V}^{(1)}$ (or, equivalently, of type $\mathbf{\Lambda}_{(1)}$).

$\mathbf{V}^{(2)}$ and $\mathbf{\Lambda}_{(2)}$ will be simply denoted by $\mathbf{V}$ and $\mathbf{\Lambda}$, respectively.

For partially ordered sets $(P, \prec)$, $(P', \prec')$, $(P'', \prec'')$ we define the *direct sum of these orders*

$$(P, \prec) = (P', \prec') \oplus (P'', \prec''),$$

as $P = P' \cup P''$ and $\prec = \prec' \cup \prec''$, if $P' \cap P'' = \emptyset$.

We say that $(P, \prec)$ is of type, for instance,

$$\mathbf{N} \oplus \mathbf{I}$$

if $(P, \prec) = (P', \prec') \oplus (P'', \prec'')$, for some $(P', \prec')$ of type $\mathbf{N}$ and $(P'', \prec'')$ of type $\mathbf{I}$. The definition extends naturally to other types, and to direct sums with a greater number of terms.

We say that a partially ordered set $(P, \prec)$ *contains*, for instance, $\mathbf{N}$, if there exists $Q \subseteq P$, such that, for some $\prec' \subseteq \prec \cap Q^2$, the partially ordered set $(Q, \prec')$ is of type $\mathbf{N}$. Again, the definition extends naturally to other types.

Let n be fixed, and let $(X, \prec)$ be a fixed partially ordered set with $|X| = n$. Let us construct a formal probability theory model rich enough to satisfy all our postulates. Let us consider three probability spaces $(\Omega_i, \mathcal{G}_i, \mathbf{Pr}_i), i = 1, 2, 3$. Let Ω_1 be the set of all permutations of X, let $\mathcal{G}_1 = \mathbf{P}(\Omega_1)$, and let $\mathbf{Pr}_1(\{\pi\}) = \frac{1}{n!}$ for each $\pi \in \Omega_1$. The only requirement we impose on $(\Omega_i, \mathcal{G}_i, \mathbf{Pr}_i), i = 2, 3$, is the existence of random variables $\overline{M} : \Omega_2 \to \{0, \dots, n\}$ of binomial distribution $Bin(n, \frac{1}{2})$ and $\overline{Z} : \Omega_3 \to \{0, 1\}$ of binomial distribution $Bin(1, \frac{1}{2})$.

Now let $\Omega = \Omega_1 \times \Omega_2 \times \Omega_3$, $\mathcal{F} = \mathcal{G}_1 \otimes \sigma(\overline{M}) \otimes \sigma(\overline{Z})$, $\mathbf{Pr} = \mathbf{Pr}_1 \otimes \mathbf{Pr}_2 \otimes \mathbf{Pr}_3$. For $(\pi, \omega, \xi) \in \Omega$, let $M(\pi, \omega, \xi) = \overline{M}(\omega)$ and $Z(\pi, \omega, \xi) = \overline{Z}(\xi)$. Obviously, M, Z and the σ-algebra $\mathcal{G}_1 \otimes \{\emptyset, \Omega_2\} \otimes \{\emptyset, \Omega_3\}$ are independent (of course, our model is finite and it is enough to talk about algebras of sets).

For $k \leq n$, let

$$\mathcal{G}_1^{(k)} = \sigma\{R_\prec : R \subseteq \{1, \ldots, k\}^2\},$$

where

$$R_\prec = \{\pi \in \Omega : \{(i,j) \in \{1, \ldots, k\}^2 : \pi(i) \prec \pi(j)\} = R\}.$$

and let $\mathcal{F}_k = \mathcal{G}_1^{(k)} \otimes \sigma(\overline{M}) \otimes \sigma(\overline{Z})$.

With the model defined above and the notation introduced we can now formally define Policy B, for short $\tau^{(n)} : \Omega \to \{1, \ldots, n\}$. For $w = (\pi, \omega, \xi) \in \Omega$, let $Y = \{\pi(1), \ldots, \pi(M(w))\}$ (note that the permutation π of the set X is also random, i.e., it is a function of w). If $Z(w) = 1$, let $T = Y_1$. If $Z(w) = 0$, let $T = Y_0$. Now, let

$$\tau^{(n)}(w) = \min\{i > M(w) : (\exists y \in T)(y \prec \pi(i)) \text{ and } (\forall j < i)(\pi(i) \not\prec \pi(j))\},$$

and $\tau^{(n)}(w) = n$ if no such $i > M(w)$ exists.

Because the equality $\tau^{(n)} = k$ depends only on Z, M and the sequence $\pi(1), \ldots, \pi(k)$, we have $[\tau^{(n)} = k] \in \mathcal{F}_k$, which means that $\tau^{(n)}$ is a stopping time with respect to the filtration $(\mathcal{F}_k)_{k \leq n}$. We omit the formal proof here.

Also note that, since T^+ is an up-set (i.e., $x \in T^+$ and $x \prec y$ implies $y \in T^+$), the above definition of $\tau^{(n)}(w)$ implies that if $T^+ \neq \emptyset$, then $\pi(\tau^{(n)}(w)) \in T^+$. Therefore the conditions $T^+ \neq \emptyset$ and $T^+ \subseteq S_1(X)$ guarantee the selector's success. It is these conditions that we will show hold sufficiently often in order to prove Theorem 3.1.

3. Main Result. The main result of this paper is the following theorem.

Theorem 3.1. *For every partially ordered set $(X, \prec)$, where $|X| = n$,*

$$\mathbf{Pr}[\pi(\tau^{(n)}) \text{ is maximal in } X] \geq \frac{1}{4}.$$

We shall prove first that, for $\tau^{(n)}$ as a universal algorithm for all partially ordered sets, the lower bound of 1/4 for $\mathbf{Pr}[\tau^{(n)} \in \mathrm{Max\,X}]$ cannot be improved.

Theorem 3.2. *There exists a sequence of partially ordered sets $(X_n, \prec_n)$ of cardinality $|X_n| = 2n+1$ such that* $\limsup_{n\to\infty} \mathbf{Pr}[\tau^{(2n+1)} \in \mathrm{Max\,X_n}] \leq 1/4$.

Proof. Let $X_n = \{x, z_1, \ldots, z_n, w_1, \ldots, w_n\}$. Let $z_i \prec x$ and $w_i \prec z_j$ for all $i, j \leq n$, and assume also that there is no relation between z_i and z_j and between w_i and w_j whenever $i \neq j$ (see Fig. 8).

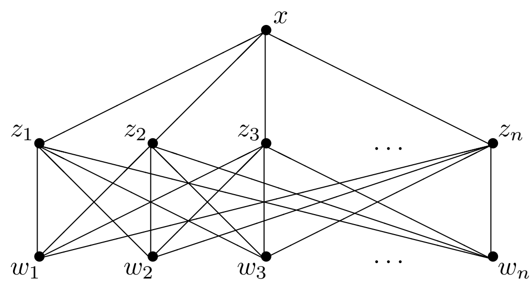

Fig. 8

The probability that there is at least one element in $\{z_1, \ldots, z_n\} \cap Y$ and in $\{w_1, \ldots, w_n\} \cap Y$ clearly tends to one with n tending to infinity.

If we are to tag elements from Y_1 ($Z = 1$), then, unless $x \in Y$, where we have no chance to make the right choice anyway, each element from $\{z_1, \ldots, z_n\} \setminus Y$ will be taken if it appears at the selector. The probability that there are more than (some fixed) m elements in $\{z_1, \ldots, z_n\} \setminus Y$ also tends to one with n tending to infinity, regardless of how big m is. This implies that with probability tending to one we shall choose one of the elements from $\{z_1, \ldots, z_n\} \setminus Y$ rather than x. Therefore we shall make a bad choice. The remaining case, where we are to tag elements from Y_0, happens with probability equal to $1/2$. The chance that x is not in Y is $1/2$, too. As these events are independent and in the former case, as we argued above, the probability of the right choice tends to zero with n tending to infinity, the chance for making the right choice from X_n asymptotically does not exceed $1/4$. This completes the proof of Theorem 3.2. $\square$

Recall that $\mathcal{P}$ is the class of ranked partially ordered sets of height 2, as defined in Section 2. In the proof of Theorem 3.1 that follows, we will consider a particular partially ordered set $(P, \prec) \in \mathcal{P}$. We will need the following series of combinatorial lemmas, which give lower bounds on the sizes of some families of subsets of P for all $(P, \prec) \in \mathcal{P}$.

For $(P, \prec) \in \mathcal{P}$, we define four families of sets:

$$\mathcal{A}_{(P,\prec)} = \{V \subseteq P : (\exists\, u, v, x, z \in P)(u \prec v \text{ and } z \prec x \text{ and } u, x, z \in V \text{ and } v \notin V)\},$$

$$\mathcal{B}_{(P,\prec)} = \{V \subseteq P : V \cap S_2(P) \neq \emptyset \text{ and } (\forall\, x, z \in P)(z \prec x \Rightarrow |\{x, z\} \cap V| \leq 1) \\ \text{and } (\forall\, x, z \in P)(z \prec x \text{ and } z \notin V, x \notin V \Rightarrow (\exists\, x' \in V)(z \prec x'))\},$$

$$\mathcal{C}_{(P,\prec)} = \{V \subseteq P : V \cap S_2(P) \neq \emptyset \text{ and } (\forall\, x, z \in P)(z \prec x \Rightarrow |\{x, z\} \cap V| \leq 1)\},$$

$$\mathcal{D}_{(P,\prec)} = \{V \subseteq P : (\exists\, x, z \in P)(z \prec x \text{ and } z \in V \text{ and } x \notin V)\}.$$

Let us notice that $\mathcal{D}_{(P,\prec)} = \mathcal{A}_{(P,\prec)} \cup \mathcal{C}_{(P,\prec)}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$, $\mathcal{A} \cap \mathcal{C} = \emptyset$.

Lemma 3.1. *If $(P,\prec) \in \mathcal{P}$ does not contain $\mathbf{N}$, $\mathbf{I} \oplus \mathbf{V}$, $\mathbf{I} \oplus \mathbf{\Lambda}$, $\mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I}$, then $(P,\prec)$ is of type $\mathbf{I} \oplus \mathbf{I}$, $\mathbf{V}^{(k)}$ or $\mathbf{\Lambda}_{(k)}$.*

Proof. Assume that P does not contain $\mathbf{N}$, $\mathbf{I} \oplus \mathbf{V}$, $\mathbf{I} \oplus \mathbf{\Lambda}$, $\mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I}$.

If $|S_1(P)| = 1$, then $(P,\prec)$ is of type $\mathbf{\Lambda}_{(k)}$.

If $|S_2(P)| = 1$, then $(P,\prec)$ is of type $\mathbf{V}^{(k)}$.

Assume now that $|S_1(P)| \geq 2$, $|S_2(P)| \geq 2$. If all elements of $S_1(P)$ dominated only one element of $S_2(P)$ we would have $|S_2(P)| = 1$. Thus there must exist x_1, x_2 of $S_1(P)$ and two different elements z_1, z_2 of $S_2(P)$ such that $z_1 \prec x_1$ and $z_2 \prec x_2$. If $x_1 \neq x_2$, then P contains $\mathbf{I} \oplus \mathbf{I}$. If $x_1 = x_2$, by the assumption that $|S_1(P)| \geq 2$ we infer that there exists $x \in S_1(P) \setminus \{x_1\}$. Then there exists $z_3 \in S_2(P)$ such that $z_3 \prec x$. Of course, $z_3 \neq z_1$ or $z_3 \neq z_2$ and we again conclude that P contains $\mathbf{I} \oplus \mathbf{I}$. Thus we can assume that P contains $\mathbf{I} \oplus \mathbf{I}$, i.e., there exist $x', x'' \in S_1(P)$ and $z', z'' \in S_2(P)$ such that $z' \prec x'$, $z'' \prec x''$, $x' \neq x''$ and $z' \neq z''$. Assume now that $|S_1(P)| > 2$. Let $x \in S_1(P) \setminus \{x', x''\}$. Let $z \in S_2(P)$ and $z \prec x$. If $z \in \{z', z''\}$ then $(P,\prec)$ contains $\mathbf{I} \oplus \mathbf{V}$, and if $z \notin \{z', z''\}$ then $(P,\prec)$ contains $\mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I}$. Both situations contradict our hypothesis. Hence $|S_1(P)| = 2$. By symmetry, also $|S_2(P)| = 2$. Thus at this moment we know that $(P,\prec)$ contains $\mathbf{I} \oplus \mathbf{I}$ and $|S_1(P)| = |S_2(P)| = 2$. Thus $(P,\prec)$ must be of type $\mathbf{I} \oplus \mathbf{I}$, for otherwise it would contain $\mathbf{N}$. This completes the proof. □

Lemma 3.2. *If $(P,\prec) \in \mathcal{P}$ contains $(P',\prec') \in \mathcal{P}$, then*

$$|\mathcal{D}_{(P,\prec)}| \geq |\mathcal{D}_{(P',\prec')}| \cdot 2^{|P|-|P'|}.$$

Proof. Let us note that if $V \in \mathcal{D}_{(P',\prec')}$ and $W \subseteq P \setminus P'$, then $V \cup W \in \mathcal{D}_{(P,\prec)}$. As there are $2^{|P|-|P'|}$ subsets W of $P \setminus P'$ the proof is complete. □

Lemma 3.3. *If $(P,\prec) \in \mathcal{P}$ contains one of the following orders $\mathbf{N}$, $\mathbf{I} \oplus \mathbf{V}$, $\mathbf{I} \oplus \mathbf{\Lambda}$, $\mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I}$, then*

$$\frac{|\mathcal{D}_{(P,\prec)}|}{2^{|P|}} \geq \frac{1}{2}.$$

Proof. It is easy to compute that if an order $(P',\prec')$ is of type $\mathbf{N}$, then $|\mathcal{D}_{(P',\prec')}| = 8$, if $(P',\prec')$ is of type $\mathbf{I} \oplus \mathbf{V}$ or of type $\mathbf{I} \oplus \mathbf{\Lambda}$, then $|\mathcal{D}_{(P',\prec')}| = 17$, and if $(P',\prec')$ is of type $\mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I}$ then $|\mathcal{D}_{(P',\prec')}| = 37$. Thus in all these cases

$$\frac{|\mathcal{D}_{(P',\prec')}|}{2^{|P'|}} \geq \frac{1}{2},$$

and the conclusion follows by the previous lemma. □

Lemma 3.4. *If $(P,\prec)$ is of type $\mathbf{V}^{(k)}$ or of type $\mathbf{\Lambda}_{(k)}$, then*

$$\frac{|\mathcal{B}_{(P,\prec)}| + |\mathcal{D}_{(P,\prec)}|}{2^{|P|}} = \frac{1}{2}.$$

Proof. The proof is a matter of a very simple counting. □

Lemma 3.5. *If* $(P, \prec)$ *is of type* $\mathbf{I} \oplus \mathbf{I}$*, then*

$$\frac{|\mathcal{B}_{(P,\prec)}| + |\mathcal{D}_{(P,\prec)}|}{2^{|P|}} = \frac{5}{8}.$$

Proof. The proof is again a matter of a very simple counting. □

In Lemmas 3.6–3.8 below we shall deal with a partially ordered set $(X, \prec)$ and any set $Y \subseteq X$. For two such sets we define the following set

$$P = \{x \in S_1(X) : (\exists z \in L(x) \cap S_2(X))(h(L(z) \cap Y) = h(Y \cap S_3(X))\}$$
$$\cup \{z \in S_2(X) : h(L(z) \cap Y) = h(Y \cap S_3(X))\}$$

(In our earlier example, with X as in Fig. 1, and $Y = \{\pi(1), \ldots, \pi(6)\}$, we get $P = \{\pi(2), \pi(6), \pi(7), \pi(9), \pi(10)\}$). It is easy to see that $(P, \prec \cap P^2) \in \mathcal{P}$, unless X is an antichain.

We shall use the following notation:

$$S_1(P) = \{x_1, \ldots, x_m\}$$

and, for $i \leq m$,

$$L(x_i) \,\cap\, P = \{z_1^{(i)}, \ldots, z_{p_i}^{(i)}\},$$

where the elements $z_1^{(i)}, \ldots, z_{p_i}^{(i)}$ are pairwise different. Finally, let $L = Y \cap S_3(X)$. Note that, by definition, $h(L) \leq h(Y) \leq h(L) + 2$; the exact value of $h(Y)$ depends on the structure of Y.

The following lemmas provide sufficient conditions for the selector's success in terms of the families of sets defined earlier.

Lemma 3.6. *Assume* $|S_1(P)| \geq 1$*. If* $P \cap Y \in \mathcal{A}_{(P,\prec \cap P^2)}$*, then for* $T = Y_1$*, we have* $T^+ \subseteq S_1(X)$ *and* $T^+ \neq \emptyset$*.*

Proof. Firstly, if $|S_1(P)| = 1$, then $\mathcal{A}_{(P,\prec \cap P^2)} = \emptyset$ and the statement is trivially true, so we can assume that $|S_1(P)| \geq 2$.

If $P \cap Y \in \mathcal{A}_{(P,\prec \cap P^2)}$ then we have $x_i \in Y, z_j^{(i)} \in Y, x_k \notin Y, z_l^{(k)} \in Y$ for some $i, k \leq m$, $j \leq p_i$, and $l \leq p_k$ (with the possibility that $z_j^{(i)} = z_l^{(k)}$). By the definition of P, we have $h_Y(z_j^{(i)}) = h_Y(z_l^{(k)}) = h(L) + 1$, $h_Y(x_i) = h(L) + 2$. Therefore $h(Y) = h(L) + 2$ and $x_i \in Y_0$, $z_j^{(i)}, z_l^{(k)} \in Y_1 = T$. So $x_k \in T^+$, which shows that $T^+ \neq \emptyset$.

Now, if $a \in T^+$ and $a \notin S_1(X)$, then there exists $b \in T = Y_1$ with $b \prec a$. Therefore $b \in S_3(X)$, so $b \in L$. But $h_Y(b) = h_Y(z_j^{(i)}) > h(L)$. This is a contradiction. □

Lemma 3.7. *Assume $|S_1(P)| \geq 1$. If $P \cap Y \in \mathcal{B}_{(P,\prec \cap P^2)}$, then $Y_0^+ \neq \emptyset$. Moreover, for $T = Y_1$, we have $T^+ \subseteq S_1(X)$ and $T^+ \neq \emptyset$.*

Proof. If $P \cap Y \in \mathcal{B}_{(P,\prec \cap P^2)}$, then we have $z_j^{(i)} \in Y$ and $x_i \notin Y$, for some $i \leq m, j \leq p_i$. We claim that $z_j^{(i)} \in Y_0$. If this is not the case, then there exists $a \in Y$ such that $h_Y(a) > h_Y(z_j^{(i)})$. Take $b \in Y$ with $b \prec a$ and $h_Y(b) = h_Y(z_j^{(i)})$. We have

$$h_Y(b) = h_Y(z_j^{(i)}) = h(L) + 1.$$

Thus $b \notin L$, whence $b \in S_2(X)$. But then $a \in S_1(X)$ and, because $h(L(b) \cap Y) = h(L)$, we have $b \in S_2(P)$ and $a \in S_1(P)$. But then $a, b \in P \cap Y$ and $b \prec a$, which contradicts the hypothesis that $P \cap Y \in \mathcal{B}_{(P,\prec \cap P^2)}$. Thus $z_j^{(i)} \in Y_0$ and $x_i \in Y_0^+$, whence $Y_0^+ \neq \emptyset$. Thus also $T^+ \neq \emptyset$ as $Y_0^+ \subseteq T^+$.

We still have to show $T^+ \subseteq S_1(X)$. Let $b \in T$ and $b \prec a$. If $a \notin S_1(X)$, then there exists $a' \in S_2(X)$ such that $a \prec a'$ or $a = a'$. Then $b \in L$, so $h_Y(b) \leq h(L)$. But $b \in T = Y_1$, and $z_j^{(i)} \in Y_0$ so

$$h_Y(b) \geq h_Y(z_j^{(i)}) - 1 = h(L(z_j^{(i)}) \cap Y) = h(S_3(X) \cap Y) = h(L).$$

Therefore $h_Y(b) = h(L)$, which implies that $a' \in P$. Now, if $a' \in Y$, then $a' \in Y_0 \subseteq T$, whence $a \notin T^+$. If $a' \notin Y$, then, as $P \cap Y \in \mathcal{B}_{(P,\prec \cap P^2)}$, we have $a' \prec x$, for some $x \in Y$. Thus $x \in Y_0 \subseteq T$, so again $a \notin T^+$. This completes the proof. □

Lemma 3.8. *Assume $|S_1(P)| \geq 1$. If $P \cap Y \in \mathcal{C}_{(P,\prec \cap P^2)}$, then for $T = Y_0$, we have $T^+ \subseteq S_1(X)$ and $T^+ \neq \emptyset$.*

Proof. If $P \cap Y \in \mathcal{C}_{(P,\prec \cap P^2)}$, then we have $z_j^{(i)} \in Y$ and $x_i \notin Y$ for some $i \leq m, j \leq p_i$. As in the previous proof, we must have $z_j^{(i)} \in Y_0 = T$, or else there exist $a, b \in P \cap Y$ with $b \prec a$, contradicting the hypothesis that $P \cap Y \in \mathcal{C}_{(P,\prec \cap P^2)}$. Therefore $x_i \in T^+$ which shows that $T^+ \neq \emptyset$.

Now, if $a \in T^+$ and $a \notin S_1(X)$, then there exists $b \in T = Y_0$ with $b \prec a$. Therefore $b \in S_3(X)$ whence also $b \in L$, but $h_Y(b) = h_Y(z_j^{(i)}) > h(L)$. This is a contradiction. □

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let Y be the following random subset of X:

$$Y = \{\pi(1), \ldots, \pi(M)\}.$$

For the sets X and Y we apply the definition of the set P.

If $|S_1(P)| = 0$, then $S_2(X) = \emptyset$. Note that in this case $\tau^{(n)} = n$ and in the last step we pick an element from $S_1(X) = X$, thus our choice is right.

Otherwise, P is some random non-empty subset of $S_1(X) \cup S_2(X)$. Let us condition on the event $P = \{u_1, \ldots, u_l\}$. We will show that conditioned on this event the probability of the selector's success is at least $1/4$, with the inequality holding for all subsets $\{u_1, \ldots, u_l\}$ of $S_1(X) \cup S_2(X)$. This is sufficient to prove the result. Let a *basic event* be an event given by a fixed membership in Y of the elements u_i, $i \leq l$, i.e., E is a basic event if E is of the form:

$$E = [u_i \in^{(i)} Y, i \leq l],$$

where $\in^{(i)}$ is either $\in$ or $\notin$, $i \leq l$. Note that there are $2^{|P|}$ basic events and the basic events are equiprobable. To see the latter, note that for fixed X, the set P depends only on the random set $Y \cap S_3(X)$, so conditioning on $P = \{u_1, \ldots, u_l\}$ does not affect the distribution of $Y \cap (S_1(X) \cup S_2(X))$; specifically, $\mathbf{Pr}[u_i \in Y] = 1/2$, for all $i \leq l$, independently. Note also that for $\mathcal{E} \subseteq \mathbf{P}(P)$ the probability $\mathbf{Pr}[P \cap Y \in \mathcal{E}]$ is equal to the number of basic events $[u_i \in^{(i)} Y, i \leq l]$ such that $\{u_i : \in^{(i)} = \in, i \leq l\} \in \mathcal{E}$ (which is, of course, $|\mathcal{E}|$) divided by the number of all basic events (which is $2^{|P|}$).

For simplicity, we shall write $(P, \prec)$ to mean $(P, \prec \cap P^2)$. If $(P, \prec)$ contains one of the following orders: $\mathbf{N}$, $\mathbf{I} \oplus \mathbf{V}$, $\mathbf{I} \oplus \mathbf{\Lambda}$ or $\mathbf{I} \oplus \mathbf{I} \oplus \mathbf{I}$, then by Lemma 3.3 we have

$$\frac{|\mathcal{D}_{(P,\prec)}|}{2^{|P|}} \geq \frac{1}{2}.$$

If $(P, \prec)$ does not contain any of the above orders, then by Lemma 3.1, $(P, \prec)$ is of type $\mathbf{I} \oplus \mathbf{I}$, $\mathbf{V}^{(k)}$ or $\mathbf{\Lambda}_{(k)}$. In this case, by Lemmas 3.4 and 3.5 we have

$$\frac{|\mathcal{B}_{(P,\prec)}| + |\mathcal{D}_{(P,\prec)}|}{2^{|P|}} \geq \frac{1}{2}.$$

Let us now estimate the probability of the selector's success. Let $Z = 1$. By Lemma 3.6, if the policy $\tau^{(n)}$ is followed, the event $[P \cap Y \in \mathcal{A}_{(P,\prec)}]$ guarantees the selector's success. By Lemma 3.7, if the policy $\tau^{(n)}$ is followed, the event $[P \cap Y \in \mathcal{B}_{(P,\prec)}]$ guarantees the selector's success. Now let $Z = 0$. By Lemma 3.8, if the policy $\tau^{(n)}$ is followed, the event $[P \cap Y \in \mathcal{C}_{(P,\prec)}]$ guarantees the selector's success.

Therefore we get

$$\begin{aligned}
\mathbf{Pr}[\tau^{(n)} \in \operatorname{Max} X] &\geq \mathbf{Pr}[Z = 1] \cdot \mathbf{Pr}([P \cap Y \in \mathcal{A}_{(P,\prec)}] \cup [P \cap Y \in \mathcal{B}_{(P,\prec)}]) \\
&\quad + \mathbf{Pr}[Z = 0] \cdot \mathbf{Pr}[P \cap Y \in \mathcal{C}_{(P,\prec)}] \\
&= \mathbf{Pr}[Z = 1] \cdot (\mathbf{Pr}[P \cap Y \in \mathcal{A}_{(P,\prec)}] + \mathbf{Pr}[P \cap Y \in \mathcal{B}_{(P,\prec)}]) \\
&\quad + \mathbf{Pr}[Z = 0] \cdot \mathbf{Pr}[P \cap Y \in \mathcal{C}_{(P,\prec)}] \\
&= \frac{1}{2} \cdot \frac{|\mathcal{A}_{(P,\prec)}| + |\mathcal{B}_{(P,\prec)}|}{2^{|P|}} + \frac{1}{2} \cdot \frac{|\mathcal{C}_{(P,\prec)}|}{2^{|P|}} \\
&= \frac{1}{2} \left(\frac{|\mathcal{D}_{(P,\prec)}|}{2^{|P|}} + \frac{|\mathcal{B}_{(P,\prec)}|}{2^{|P|}} \right) \geq \frac{1}{4}. \qquad \square
\end{aligned}$$

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