Markov chains with heavy-tailed increments and asymptotically zero drift

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Joint work with Mikhail Menshikov, Dimitri Petritis and Andrew Wade

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Markov chains on $\mathbb{R}_+$ "with 2 moments"

What determines the asymptotics of a Markov chain (X_n) on $\mathbb{R}_+$?

Recurrence vs. transience

recurrence: $\exists\, x_0 < \infty$ s.t. $\liminf_{n\to\infty} X_n \le x_0$, a.s.

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Lamperti (1960): asymptotics given by relative sizes of μ_1, μ_2, where

$$\mu_k(x) := \mathbb{E}[(X_{n+1} - X_n)^k \mid X_n = x].$$

Critical regime has $\mu_1(x) \sim \frac{b}{x}$, $\mu_2(x) \sim c$:

- $2b \leq c$ implies recurrence;
- $2b > c$ implies transience.

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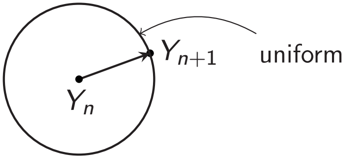
Example: Pearson–Rayleigh Random Walk on $\mathbb{R}^d$

Random walk (Y_n) with jump $Y_{n+1} - Y_n$ uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$.



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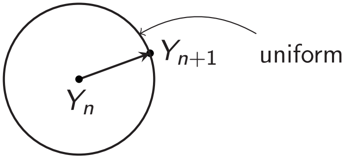

Then $X_n = \|Y_n\|$ is a Markov chain on $\mathbb{R}_+$ with

$$\mu_1(x) \sim \left(1 - \frac{1}{d}\right)\frac{1}{2x}, \quad \mu_2(x) \sim \frac{1}{d}.$$

Hence, recurrent if $d \leq 2$; transient if $d > 2$.

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What if $\mu_2(x) = \infty$?

$$\mu_k(x) := \mathbb{E}[(X_{n+1} - X_n)^k \mid X_n = x]$$

Suppose

$$\mathbb{P}[X_{n+1} - X_n > y \mid X_n = x] \sim c y^{-\alpha},$$

(uniform in x, as $y \to \infty$), for constants $\alpha \in (1, 2)$ and $c > 0$.
Then $\mu_2(x) = \infty$, but $\mu_1(x)$ is finite.

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Notation

- $\mathbb{P}_x[\,\cdot\,] := \mathbb{P}[\,\cdot \mid X_0 = x]$, $\quad \mathbb{E}_x[\,\cdot\,] := \mathbb{E}[\,\cdot \mid X_0 = x]$,
- $\Delta := X_1 - X_0$,
- $\Delta_+ := \max\{\Delta, 0\}$ (right jumps),
- $\Delta_- := -\min\{\Delta, 0\}$ (left jumps).

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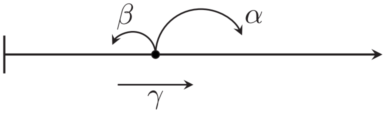
Heavy-tailed Markov chains on $\mathbb{R}_+$

Assumptions

- $\mathbb{P}_x[\Delta > y] \sim cy^{-\alpha}$ for some $\alpha \in (1, 2)$ and $c > 0$,
- $\mathbb{E}_x[\Delta_-^\beta] < \infty$ for some $\beta > \alpha$,
- $\mathbb{E}_x[\Delta] \sim bx^{-\gamma}$ for some $\gamma \geq 0$ and $b \in \mathbb{R}$.



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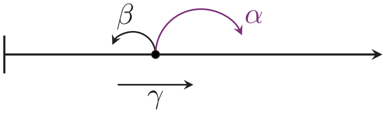

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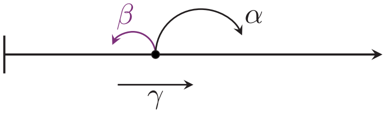

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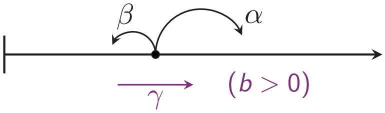

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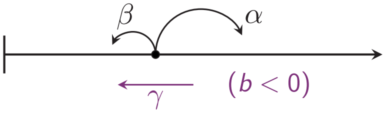
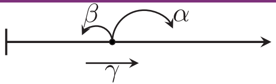


Recurrence classification



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Theorem (GMPW, 2019)

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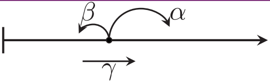

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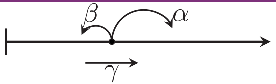

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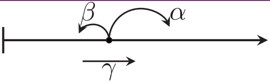

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Heavy-tailed martingale on $\mathbb{R}_+$: how recurrent?

We can quantify recurrence by determining which moments of passage times exist:

For $a > 0$, define $\tau_a := \min\{n \geq 0 : X_n \leq a\}$. Then, there exists $a > 0$ such that for all $x > a$:

$$\mathbb{E}_x[\tau_a^q] < \infty \text{ for } q < 1/\alpha,$$
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Compare with simple symmetric random walk on $\mathbb{R}_+$:

$$\mathbb{E}_x[\tau_a^q] < \infty \text{ for } q < 1/2,$$
$$\mathbb{E}_x[\tau_a^q] = \infty \text{ for } q > 1/2.$$

In this sense, our heavy-tailed martingale is more recurrent than SSRW (but is still null recurrent).

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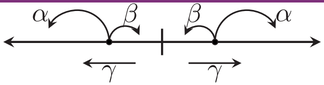
MCs on $\mathbb{R}$: heavier outwards tails



$$x \geq 0: \quad \mathbb{P}_x[\Delta_+ > y] \sim cy^{-\alpha}, \quad \mathbb{E}_x[\Delta_-^{\beta}] < \infty, \quad \mathbb{E}_x[\Delta] \sim bx^{-\gamma}$$

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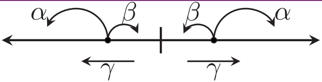

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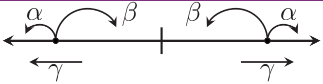

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Recurrence/transience on $\mathbb{R}$

On state space $\mathbb{R}$:

- (X_n) is recurrent if $\exists\, x_0 < \infty$ s.t. $\liminf_{n\to\infty} |X_n| \le x_0$, a.s.,
- (X_n) is transient if $\lim_{n\to\infty} |X_n| = \infty$, a.s.

Two types of transience

directional transience: $\lim_{n\to\infty} X_n = \pm\infty$, a.s.,

oscillatory transience: $\lim_{n\to\infty} |X_n| = \infty$ and $-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty$, a.s.

For chains with heavier outward tails we can only have directional transience. For chains with heavier inwards tail we can also have oscillatory transience.

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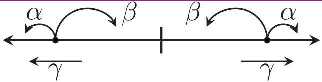
MCs on $\mathbb{R}$: heavier inwards tails



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Suppose (X_n) on $\mathbb{R}$ satisfies our assumptions. Then,

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  if $b > 0$ then (X_n) is directional transient

- *critical drift:* *if $\gamma = \beta - 1$ then*

  if $b + c\pi \cot(\pi\beta) < 0$ then (X_n) is recurrent
  if $b + c\pi \cot(\pi\beta) > 0$ then (X_n) is oscillatory transient

- *small drift:* *if $\gamma > \beta - 1$*

  if $\beta > 3/2$ then (X_n) is recurrent
  if $\beta < 3/2$ then (X_n) is oscillatory transient

Some surprising(?) behaviours

(X_n) : 

- there exist martingales on $\mathbb{R}$ with heavy tails away from the origin that are more recurrent than SSRW on $\mathbb{R}$.

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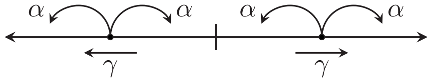
(X_n) :

- there exist martingales on $\mathbb{R}$ with heavy tails towards the origin (exponent $\beta < 3/2$) that are transient.

Other heavy-tailed Markov chains

Our analysis also covers the balanced case ($\alpha = \beta$):



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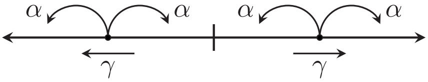
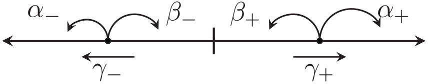


With more work, our techniques should also allow for general exponents $\alpha_\pm, \beta_\pm, \gamma_\pm$:



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References

Georgiou, N., Menshikov, M. V., Petritis, D. and Wade, A. R., Markov chains with heavy-tailed increments and asymptotically zero drift, Electron. J. Probab., **24** (2019), no. 62, 1–28.

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Thanks for your attention!

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